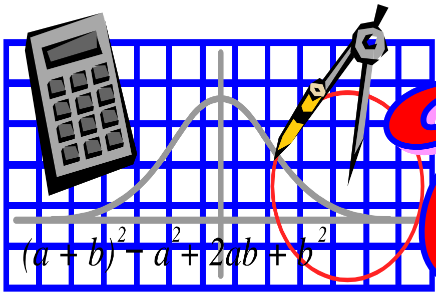
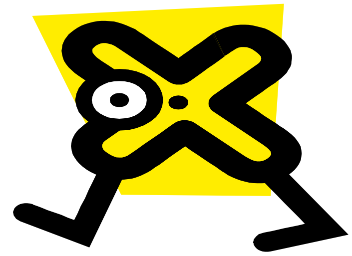


Algebra Connections

Mr. Breitsprecher's Edition March 3, 2005 Web: www.clubtnt.org/my_algebra

Factoring Polynomials



Factoring a polynomial is the opposite process of multiplying polynomials. Factor means write as a product. Recall that when we factor a number, we are looking for prime factors that multiply together to give the number; for example

$$6 = 2 * 3, \text{ or } 12 = 2 * 2 * 3.$$

When we factor a polynomial, we are looking for simpler polynomials that can be multiplied together to give us the polynomial that we started with. Understanding how to multiply polynomials is the key to understanding how to factor them.

When we factor a polynomial, we are usually only interested in breaking it down into polynomials that have integer coefficients and constants.

Simplest Case: Removing Common Factors

The simplest type of factoring is when there is a factor common to every term. In that case, you can factor out that common factor. What you are doing is using the distributive property in reverse.

Recall that the distributive law says

$$a(b + c) = ab + ac.$$

Source: www.jamesbrennan.org/algebra

Thinking about it in reverse means that if you see ab + ac, you can write it as a(b + c).

Example: $2x^2 + 4x$

Notice that each term has a factor of 2x, so we can rewrite it as:

$$2x^2 + 4x = 2x(x + 2)$$

Difference of Two Squares

If you see something of the form $(a^2 - b^2)$, you should remember the formula

$$(a+b)(a-b) = a^2-b^2$$

Example: $x^2 – 4 = (x – 2)(x + 2)$

This only holds for a **DIFFERENCE** of two squares. There is no way to factor a sum of two squares such as $a^2 + b^2$ into factors with real numbers. Trying to do so is a common mistake in algebra – please avoid this one!

Trinomials (Quadratic)

A quadratic trinomial has the form

$$ax^2 + bx + c,$$

where the coefficients a, b, and c, are real numbers (for simplicity we will only use integers, but in real life they could be any real number). We are interested here in factoring quadratic trinomials with integer coefficients into factors that have integer coefficients.

Not all such quadratic polynomials can be factored using the real numbers, and even fewer into integers. Therefore, when we say a quadratic can be factored, we mean that we can write the factors with only integer coefficients.

If a quadratic can be factored, it will be the product of two first-degree binomials, except for very simple cases that just involve monomials. For example x^2 by itself is a quadratic expression where the coefficient a is equal to 1, and b and c are zero. We know that x^2 factors into (x)(x).

Another situation occurs when only the coefficient c is zero. Then you get something that looks like

$$2x^2 + 3x$$

This can be factored very simply by factoring out ('undistributing') the common factor of x:

$$2x^2 + 3x = x(2x + 3)$$

The most general case is when all three terms are present, as in

$$x^2 + 5x + 6$$

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We look at two cases of this type. The easiest to factor are the ones where the coefficient of x^2 (which we are calling 'a') is equal to 1, as in the above example. This is the simplest case; let's begin by looking at a = 1 examples.

Coefficient of x^2 is 1

Since the trinomial comes from multiplying two first-degree binomials, let's review what happens when we multiply binomials using the **FOIL** method. Remember that to do factoring we will have to think about this process in reverse (you could say we want to 'de-FOIL' the trinomial).

Suppose we are given

$$(x + 2)(x + 3)$$

Using the FOIL method, we get

$$(x + 2)(x + 3) = x2 + 3x + 2x + 6$$

Then, collecting like terms gives

$$(x + 2)(x + 3) = x2 + 5x + 6$$

Notice where the terms in the trinomial came from. The x^2 came from x times x. The interesting part is what happens with the other parts, the '+ 2' and the '+ 3'.

The last term in the trinomial, the 6 in this case, came from multiplying the 2 and the 3. Where did the 5x in the middle come from? We got the 5x by adding the 2x and the 3x when we collected like terms. We can state this as a rule:

If the coefficient of x^2 is one, then to factor the quadratic you need to find two numbers that:

1. Multiply to give the constant term (which we call c)
2. Add to give the coefficient of x (which we call b)

This rule works even if there are minus signs in the quadratic expression, if we remember how to add/subtract and multiply/divide positive and negative numbers.

Special Case: Perfect Square Trinomial

Recall from special products of binomials :

- $(a+b)^2 = a^2+2ab+b2$
- $(a-b)^2 = a^2 – 2ab + b^2$

The trinomials on the right are called perfect squares because they are the squares of a single binomial, rather than the product of two different binomials. A quadratic trinomial can also have this form:

$$(x + 3)2 = (x + 3)(x + 3) = x^2 + 6x + 9$$

Notice that the coefficient of x is the sum 3 + 3, and the constant term is the product 3 * 3. One can also say:

- The coefficient of x is twice the number 3
- The constant term is the number three squared

In general, if a quadratic trinomial is a perfect square, then

- The coefficient of x is twice the square root of the constant term

Or to restate it another way,

- The constant term is the square of half the coefficient of x

In symbolic form, we can express this as:

$$(x+a)^2 = x^2+2ax+a^2$$

It is important to be able to recognize perfect square trinomials. This is the key to solving quadratic equations.

Coefficient of x^2 is not 1

A quadratic is more difficult to factor when the coefficient of the squared term is not 1, because that coefficient is mixed in with the other products from FOILing the two binomials. There are two methods for attacking these: either you can use a systematic guess-and-check method, or a method called factoring by grouping. We will first look at the guess-and-check method.

If you need to factor a trinomial such as

$$2x^2 + x - 3$$

Think about what combinations could give the $2x^2$ as well as the other two terms.

In this example the $2x^2$ must come from (x)(2x), and the constant term might come from either (-1)(3) or (1)(-3). Now we must figure out which combination will give the correct middle term. By "trial and error" we get:

$$(x-1)(2x+3)$$

Notice these patterns.

- The first term in the trinomial (the $2x^2$) is just the product of the leading terms in the binomials.
- The constant term in the trinomial (the -3) is the product of the constant terms in the binomials (so far this is the same as in the case where the coefficient of x^2 is 1)
- The middle term in the trinomial (the x) is the sum of the outer and inner products. This involves all the constants and coefficients and is not always obvious.

Because 1 and 2 are relatively simple and 3 is complicated, it makes sense to think of the possible candidates that would satisfy conditions 1 and 2, and then test them in every possible combination by multiplying the resulting binomials to see if you get the correct middle term.

Step-by-Step Process

1. List all the possible ways to get the coefficient of x^2 (which we call "a") by multiplying two numbers
2. List all the possible ways to get the constant term (which we call "c") by multiplying two numbers
3. Try all possible combinations of these to see which ones give the correct middle term
   - Don't forget that the number itself times 1 is a possibility
   - If the number (a or c) is negative, remember to try the plus and minus signs both ways

In our example: $2x^2 + x – 3$, we make a list of the possible factors of $2x^2$: The only choice is (2x)(x).

Then we make a list of the possible factors of the constant term -3: it is either (1)(-3) or (-1)(3).

The possible factors of the trinomial are the binomials that we can make out of these possible factors, taken in every possible order.

$$(2x + 1)(x – 3)$$
$$(x + 1)(2x – 3)$$
$$(2x + 3)(x – 1)$$
$$(x + 3)(2x – 1)$$

Multiplying these and find the one that works (the third one). **All you really need to check is to see if the sum of the outer and inner multiplications will give you the correct middle term**, since we already know that we will get the correct first and last terms.